PERIODIC CONJUGATE NETS

A DISSERTATION

PRESENTED TO THE

FACULTY OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

BY

EDWARD S. HAMMOND

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BY EDWARD S. HAMMOND.

Introduction. If n functions of u and v, $x^{(1)}$, $x^{(2)}$, $\cdots$, $x^{(n)}$, which satisfy an equation of Laplace of the form

$$\text{(1)} \qquad \frac{\partial^2 x}{\partial u \partial v} = \frac{\partial \log a}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial x}{\partial v} + cx,$$

be interpreted as the homogeneous coördinates of a surface in $(n-1)$ space, the parametric curves on this surface are said to form a *conjugate net*. Where no ambiguity arises, this system of curves or the surface on which it lies will be called simply the net N. Equation (1) will be called the *point equation* of N. Now the functions* x_1 and x_{-1}, given by

$$\text{(2)} \qquad x_1 = \frac{\partial x}{\partial v} - \frac{\partial \log a}{\partial v}x, \qquad x_{-1} = \frac{\partial x}{\partial u} - \frac{\partial \log b}{\partial u}x,$$

are also homogeneous coördinates of nets, N_1 and N_{-1}, which are called the *first* and *minus first Laplace transforms of* N. N_1 has as its first and minus first Laplace transforms nets N_2 and N itself; N_2 is called the second Laplace transform of N. Developing these transforms in both senses we get a series of nets $\cdots N_{-s}, \cdots, N_{-1}, N, N_1, \cdots, N_r, \cdots,$ called a sequence of Laplace.† This sequence will be called the sequence N_r. In the first section of this paper general properties of this sequence will be developed.

In section 2, we impose upon the sequence N_r the condition that it shall be periodic; that is, that a certain Laplace transform N_p of N shall coincide with N itself. After transformation of parameters it is shown that the identity of N_p and N involves the identity of N_{p-1} and N_{-1} and in general, of N_{p-k} and N_{-k}, $k = 0, 1, 2, \cdots, p$. Necessary conditions on the coefficients of the point equation of N are derived and it is shown by discussion of the completely integrable systems of partial differential equations involved that these conditions are also sufficient. It is also shown that if an equation of Laplace of form (1) is the point equation of one periodic net, it is the point equation of an infinity of others of the same period.

The remainder of the paper is taken up with other sequences of Laplace

* Here x_1 indicates any or all of $x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}$. A similar usage is followed throughout.

† Darboux, Leçons sur la théorie générale des surfaces, 2d ed. (1915), vol. II, chap. 2.

closely related to the sequence N_r. The sequences studied in section 3 involve certain properties of families of lines in higher ordered spaces which we proceed to develop. The lines joining corresponding points of a net N and its first Laplace transform N_1 form a two-parameter family G, each line of which is a common tangent of these surfaces. Consider for the sake of definiteness the line joining the points on N and N_1 with parameters u_0 and v_0. Through this line pass two developable surfaces all of whose generators are lines of G, namely, the tangent surfaces of the curve $u = u_0$ on N, and of the curve $v = v_0$ on N_1. When a two-parameter family of lines in higher ordered space possesses either of these equivalent properties, namely, that each line of the family is a common tangent to two surfaces, and that through each line there pass two developable surfaces all of whose rectilinear generators are lines of the family, it is called a *congruence*. In 3-space any two parameter family of lines possesses these properties, but in space of higher order this is not the case. The surfaces to which all the lines of G are tangent are called the *focal surfaces* and the nets N and N_1 the *focal nets* of the congruence.

Levy* has shown that the functions ξ and η, defined by

$$\xi = x - \frac{\theta}{\dfrac{\partial \theta}{\partial v}} \frac{\partial x}{\partial v}, \qquad \eta = x - \frac{\theta}{\dfrac{\partial \theta}{\partial u}} \frac{\partial x}{\partial u}, \tag{3}$$

where θ is any solution of (1), may be interpreted as the coördinates of nets which will be called *Levy transforms* of N by means of θ. The points of these nets lie on the lines joining the corresponding points of N and N_1, N_{-1} and N, respectively, and the developables of the congruences so generated cut the surfaces of the nets in the curves of the nets. In section 3, it is shown that these nets, there called $N_{0,1}$ and $N_{-1,0}$, are Laplace transforms of one another. It is also shown that $N_{0,1}$ is a Levy transform of N_1 by means of θ_1, a solution of the point equation of N_1 formed from θ by the same process by which the coördinates of N_1 were formed from those of N. From these properties follows a very intimate connection between the two sequences of Laplace, N_r, the original sequence, and $N_{r,r+1}$ of which $N_{-1,0}$ and $N_{0,1}$ are two nets. The sequence $N_{r,r+1}$ is called the *first Levy sequence*. On it may be formed a first Levy sequence, $N_{r,r+2}$ which is called a *second Levy sequence* of N. The treatment given in section 4 of these sequences and of the Levy sequences of higher orders which are analogously formed, indicates their close dependence upon the Laplace transforms of N. They are actually the sequences of derived nets of higher orders studied by Tzitzeica† and others.

* Levy, Journal de l'École Polytechnique, Vol. LVI (1886), p. 67.

† Tzitzeica, Comptes Rendus, vol. 156 (1913), p. 375.

In section 5, the results of section 2 are applied to these Levy sequences and conditions for their periodicity are derived. Two interesting geometric configurations arising under special conditions are discussed.

As a property of the Levy transforms of a net N, it was mentioned that the developables of the congruences of tangents to the parametric curves of N cut the surfaces of the Levy transforms in the curves of the nets. Whenever this relation holds between a congruence and a net, they are said to be *conjugate.* Two nets conjugate to the same congruence are said to be *in relation* T and the transformation which carries one such net into the other is called a *transformation* T, to use the terminology of Eisenhart* who has developed a general theory of such transformations. The congruence to which both nets are conjugate is called the *conjugate congruence* of the transformation. In section 6, it is shown that similar Laplace transforms of two nets in relation T are also in relation T, and hence that two sequences of Laplace may be developed such that corresponding nets of these sequences are in relation T. The problem of finding a sequence $\bar{N}_r$ so related to the original sequence N_r is reduced to the problem of finding a solution ϕ of the adjoint equation of (1) and quadratures. Owing to arbitrary constants arising in the quadratures, their integration gives a multiple infinity of such sequences between which certain geometric relations exist.

The results of section 2 are then applied to these sequences, and it is found, first, that if equation (1) has periodic solutions, so has its adjoint; second, if such a solution ϕ be used in the determination of $\bar{N}_r$, the sequences $\bar{N}_r$ are also periodic of period p.

1. Sequences of Laplace. In the study of these sequences, two functions of the coefficients of equation (1), H and K, defined by

$$(4)\qquad \begin{aligned} H &= -\frac{\partial^2 \log a}{\partial u \partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c, \\ K &= -\frac{\partial^2 \log b}{\partial u \partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c, \end{aligned}$$

are of constant occurrence. Their most important property is in connection with the transformation to other coördinates x', such that

$$(5)\qquad x = \lambda x',$$

where λ is a function of u and v. Since the coördinates x are homogeneous, evidently this transformation has no effect on the net. The coördinates x' do not satisfy equation (1), however, but are solutions of

* Eisenhart, Trans. Amer. Math. Soc., vol. 18 (1917), p. 97.

$$
(6)\quad
\begin{aligned}
\frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial}{\partial v}\log\frac{a}{\lambda}\frac{\partial\theta}{\partial u} + \frac{\partial}{\partial u}\log\frac{b}{\lambda}\frac{\partial\theta}{\partial v} \\
&\quad + \left(-\frac{\partial^2}{\partial u\partial v}\log\lambda - \frac{\partial}{\partial v}\log\frac{a}{\lambda}\frac{\partial}{\partial u}\log\frac{b}{\lambda} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c\right)\theta,
\end{aligned}
$$

as may be shown by differentiation. If the functions H and K be formed from the coefficients of (6) and the resulting expressions reduced, it is found that they are identical with (4), that is, H and K are invariant under the transformation (5). They are called the *Laplace-Darboux invariants* of the equation (1) or of the net N. If the independent variables are changed by a transformation

$$
(7)\qquad u = \phi(u'), \qquad v = \psi(v'),
$$

the invariants H' and K' of the new equation are given by

$$
(8)\qquad H' = \phi'(u')\psi'(v')H, \qquad K' = \phi'(u')\psi'(v')K,
$$

where ϕ' and ψ' are the first derivatives of ϕ and ψ with respect to their arguments.

Consider the coördinates

$$
(9)\qquad x_1 = \frac{\partial x}{\partial v} - \frac{\partial\log a}{\partial v}x,
$$

of the net N_1 mentioned in the introduction. If we differentiate with respect to u, we get

$$
(10)\qquad \frac{\partial x_1}{\partial u} - \frac{\partial\log b}{\partial u}x_1 = Hx,
$$

a relation confirming the statement of the introduction that the lines joining corresponding points of N and N_1 are tangent to the curves $v =$ const. on N_1. Then if H vanishes equations (9) and (10) reduce the solution of equation (1) to quadratures; also in this case, the surface N_1 degenerates into a curve. But if H does not vanish, we differentiate with respect to v and find that the coördinates x_1 satisfy the equation of Laplace

$$
(11)\quad
\begin{aligned}
\frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial\log aH}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial\log b}{\partial u}\frac{\partial\theta}{\partial v} \\
&\quad + \left(\frac{\partial^2}{\partial u\partial v}\log\frac{b}{a} - \frac{\partial\log aH}{\partial v}\frac{\partial\log b}{\partial u} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c\right)\theta,
\end{aligned}
$$

which proves that N_1 is also a net, as stated in the introduction. This equation has invariants H_1 and K_1, analogous to H and K, defined by

$$
(12)\quad
\begin{aligned}
H_1 &= -\frac{\partial^2}{\partial u\partial v}\log\frac{a^2H}{b} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c = H - \frac{\partial^2}{\partial u\partial v}\log\frac{aH}{b}, \\
K_1 &= -\frac{\partial^2}{\partial u\partial v}\log a + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c = H.
\end{aligned}
$$

Since frequent use is to be made of the point equation of nets associated with a net having the point equation (1), for the sake of brevity we denote such an equation by the expression

$$(13) \qquad [x_i;\ a_i, b_i, c_i],$$

which means that the coördinates x_i of the net N_i satisfy the equation

$$(14) \qquad \begin{aligned}\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial \log a_i}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b_i}{\partial u}\frac{\partial \theta}{\partial v} \\ + \left(\frac{\partial^2 \log c_i}{\partial u\partial v} - \frac{\partial \log a_i}{\partial v}\frac{\partial \log b_i}{\partial u} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c\right)\theta.\end{aligned}$$

Also the net N_i has invariants

$$(15) \qquad \begin{aligned} H_i &= -\frac{\partial^2}{\partial u\partial v}\log\frac{a_i}{c_i} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c, \\ K_i &= -\frac{\partial^2}{\partial u\partial v}\log\frac{b_i}{c_i} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c.\end{aligned}$$

In this notation (11) becomes

$$(16) \qquad [x_1;\ aH, b, b/a],$$

and the effect of the transformation (5) on the point equation is expressed by

$$(17) \qquad \left[\frac{x_i}{\lambda};\ \frac{a_i}{\lambda}, \frac{b_i}{\lambda}, \frac{c_i}{\lambda}\right].$$

The minus first Laplace transform, N_{-1}, is the second focal surface of the congruence of tangents to the curves $v =$ const. of N. For, by the definition of its coördinates given in equation (2), the lines joining corresponding points are tangent to N, and the equation obtained by differentiating these coördinates with respect to v and using (1) shows them to be tangent to N_{-1}. The point equation of this net is denoted by

$$(18) \qquad [x_{-1};\ a, bK, a/b].$$

Consider now the congruences of tangents to the parametric curves of N_1. We have seen that the congruence of tangents to the curves $v =$ const. has N and N_1 as its focal nets; that is, N is the minus first Laplace transform of N_1. This is also obvious as a consequence of equation (10), whose left member is the expression for the coördinates of the minus first Laplace transform of N_1 formed by analogy with (2).

The second focal surface of the congruence of tangents to the curves $u =$ const. of N_1 is the net N_2, the second Laplace transform of N. Its

coördinates

$$x_2 = \frac{\partial x_1}{\partial v} - \frac{\partial \log aH}{\partial v} x_1, \tag{19}$$

or, using (9),

$$x_2 = \frac{\partial^2 x}{\partial v^2} - \frac{\partial \log a^2 H}{\partial v}\frac{\partial x}{\partial v} + \left(\frac{\partial \log a}{\partial v}\frac{\partial \log aH}{\partial v} - \frac{\partial^2 \log a}{\partial v^2}\right) x,$$

satisfy the equation denoted by $[x_2;\ aHH_1,\ b,\ b^2/a^2H]$.

Continuation of this process in both the positive and negative senses gives the nets of the sequence N_r.

The coördinates of the rth Laplace transform N_r are

$$x_r = \frac{\partial x_{r-1}}{\partial v} - \frac{\partial \log aHH_1 \cdots H_{r-2}}{\partial v} x_{r-1}, \tag{20}$$

or, by repeated substitution,

$$x_r = \frac{\partial^r x}{\partial v^r} + A_{r,\,r-1}\frac{\partial^{r-1} x}{\partial v^{r-1}} + \cdots + A_{r,\,0}x. \tag{21}$$

where the $A_{p,\,q}$ are functions of a, H, H_1, $\cdots$, H_{r-2} and their derivatives. The point equation of N_r is

$$\left[x_r;\ aHH_1 \cdots H_{r-1},\ b,\ \frac{b^r}{a^r H^{r-1} H_1^{r-2} \cdots H_{r-2}}\right]. \tag{22}$$

The equations analogous to (10) and (19) are

$$\frac{\partial x_r}{\partial u} = H_{r-1}x_{r-1} + \frac{\partial \log b}{\partial u} x_r; \qquad \frac{\partial x_r}{\partial v} = \frac{\partial \log a_r}{\partial v} x_r + x_{r+1}, \tag{23}$$

and they will be used as formulas for the partial derivatives $\partial x_r/\partial u$ and $\partial x_r/\partial v$.

On the negative side of the sequence, the general Laplace transform N_{-s} has coördinates

$$x_{-s} = \frac{\partial x_{-s+1}}{\partial u} - \frac{\partial \log bKK_{-1} \cdots K_{-s+2}}{\partial u} x_{-s+1},$$

or

$$x_{-s} = \frac{\partial^s x}{\partial u^s} + B_{s,\,s-1}\frac{\partial^{s-1} x}{\partial u^{s-1}} + \cdots + B_{s,\,1}\frac{\partial x}{\partial u} + B_{s,\,0}x, \tag{24}$$

which satisfy the equation

$$\left[x_{-s};\ a, bKK_{-1} \cdots K_{-s+1},\ \frac{a^s}{b^s K^{s-1} \cdots K_{-s+2}}\right]. \tag{25}$$

The formulas corresponding to (23) are

$$\frac{\partial x_{-s}}{\partial u} = x_{-s-1} + \frac{\partial \log b_{-s}}{\partial u} x_{-s}; \qquad \frac{\partial x_{-s}}{\partial v} = \frac{\partial \log a}{\partial v} x_{-s} + K_{-s+1}x_{-s+1}. \tag{26}$$

From (23) and (26) it follows that if H_{r-1} or K_{-s+1} vanishes, the sequence terminates; for the surface N_r or N_{-s} degenerates into a curve. This is a special case of great importance* but it is not before us in this paper.

2. **Periodic Sequences of Laplace.** In the introduction, a periodic sequence was defined as a sequence such that a certain net N_p coincided with the original net N. When this is the case, the coördinates x_p and x must satisfy the relation

$$x_p = \lambda(u, v)x, \tag{27}$$

where λ is a function of u and v at most, and is the same for all n coördinates. The coördinates x_p satisfy the equation denoted by

$$\left[x_p;\ aHH_1 \cdots H_{p-1},\ b,\ \frac{b^p}{a^pH^{p-1}\cdots H_{p-2}}\right], \tag{28}$$

this result being obtained when r in (22) is replaced by p. From (17), (27), and (28), the coördinates x must satisfy

$$\left[x;\ \frac{aHH_1 \cdots H_{p-1}}{\lambda},\ \frac{b}{\lambda},\ \frac{b^p}{a^pH^{p-1}\cdots H_{p-2}\lambda}\right], \tag{29}$$

as well as the fundamental equation (1). Since in every case which we shall consider there are at least three coördinates x, the coefficients of $\partial x/\partial u$, $\partial x/\partial v$, and x in (29) and in (1) must be equal. We have therefore

$$\frac{\partial}{\partial v}\log\frac{aH \cdots H_{p-1}}{\lambda} = \frac{\partial \log a}{\partial v}, \qquad \frac{\partial}{\partial u}\log\frac{b}{\lambda} = \frac{\partial \log b}{\partial u}, \tag{30}$$

$$\frac{\partial^2}{\partial u\partial v}\log\frac{b^p}{a^pH^{p-1}\cdots H_{p-2}\lambda} = 0. \tag{31}$$

From the equations (30), we get

$$\frac{\partial \log \lambda}{\partial v} = \frac{\partial \log HH_1 \cdots H_{p-1}}{\partial v}, \qquad \frac{\partial \log \lambda}{\partial u} = 0, \tag{32}$$

and from these

$$\frac{\partial^2}{\partial u\partial v}\log HH_1 \cdots H_{p-1} = 0. \tag{33}$$

Using (32) and (33) in (31), we get

$$\frac{\partial^2}{\partial u\partial v}\log\frac{b^p}{a^pH^{p-1}\cdots H_{p-2}} = 0, \tag{34}$$

which can also be obtained immediately from the equality of H, the invariant of (1), and H_p, the corresponding invariant of (28).

* Darboux, l.c., p. 33.

Equation (33) may be integrated, giving

$$HH_1 \cdots H_{p-1} = UV,$$

where U and V are functions of u and v alone respectively.

From equation (8) we recall the effect of the transformation (7) on the invariants H and K. Likewise under this same transformation

$$H_i' = \phi'(u')\psi'(v')H_i.$$

By giving to i values from 0 to $p - 1$ and multiplying, we get

$$H'H_1' \cdots H_{p-1}' = HH_1 \cdots H_{p-1}[\phi'\psi']^p = \bar{U}(u')\bar{V}(v')[\phi'\psi']^p,$$

where $\bar{U}$ and $\bar{V}$ are the transforms of U and V under (7). Hence ϕ and ψ may be determined so that

$$H'H_1' \cdots H_{p-1}' = 1;$$

then from (32), λ equals a constant,* m, since

$$\frac{\partial \log \lambda}{\partial u} = \frac{\partial \log \lambda}{\partial v} = 0.$$

In the remainder of this section, we shall assume that this transformation has been made, dropping primes for convenience.

After this change of variable, there are two necessary conditions for a sequence of Laplace of period p, namely

$$HH_1H_2 \cdots H_{p-1} = 1 \tag{36}$$

and equation (34). To show that these conditions are sufficient, we proceed as follows. Differentiate

$$x_p = mx \tag{37}$$

with respect to u. Using (23), (2), and (37), we find

$$H_{p-1}x_{p-1} = mx_{-1}, \tag{38}$$

which states analytically the fact, evident from geometry, that if X_p coincides with X, then X_{p-1} coincides with X_{-1}. Differentiating the last equation with respect to u, and using (23) and (26), we have

$$H_{p-1}H_{p-2}x_{p-2} + H_{p-1}\frac{\partial \log bH_{p-1}}{\partial u}x_{p-1} = mx_{-2} + m\frac{\partial \log bK}{\partial u}x_{-1}. \tag{39}$$

Now $K = H_{-1}$ and $H_{-1} = H_{p-1}$, since (38) is a transformation of the type (5). The equality of K and H_{p-1} may also be derived from (34) and (36), using the values of these invariants given by (15). Then by (38),

* Tzitzeica, Comptes Rendus, vol. 157 (1913), p. 908.

equation (39) reduces to

$$(40) \qquad H_{p-1}H_{p-2}x_{p-2} = mx_{-2}.$$

If we continue this process we have in general,

$$(41) \qquad H_{p-1}H_{p-2} \cdots H_{p-i}x_{p-i} = mx_{-i},$$

or, by (36),

$$(42) \qquad x_{p-i} = mHH_1 \cdots H_{p-i-1}x_{-i}$$

and finally

$$(43) \qquad x = mx_{-p}$$

showing that N is identical with its minus pth Laplace transform as well as with the pth transform. We observe that this process is reversible, that is, by starting from (43), differentiating with respect to v, and reducing step by step, we may reproduce this same set of equations.

If we refer to (21) and (24) it is evident that the $p + 1$ equations given by (41) when i takes integral values from 0 to p inclusive, are a system of linear partial differential equations of various orders which, with (1), must be satisfied by the coördinates of the fundamental net N of a periodic sequence. From this point of view, let us examine in detail the transformation from equation (38) into (40), as this is entirely typical of the change from any one of (41) into the next. The substitutions for $\partial x_{p-1}/\partial u$ and $\partial x_{-1}/\partial u$ from (23) and (2) first engage our attention. The value of $\partial x_{p-1}/\partial u$ used depends on the definition of x_{p-2} and on the use of the point equation of N_{p-2}. But this point equation is essentially the result of differentiating (1) $p - 2$ times with respect to v, a fact which becomes evident on consideration of the result of substituting the value of x_{p-2} from (21) in the point equation denoted by (22). The value of $\partial x_{-1}/\partial u$ used is merely the definition of the minus first Laplace transform. The rest of the reduction may be based as indicated on the two equations (34) and (36). From these considerations and from the reversibility of the process we conclude that, by virtue of (1), its derivatives, and the conditions (34) and (36), any one of equations (41) or (42) is equivalent to any of the others.

If the period be odd, let us set $p = 2n + 1$, and $i = n$ in (42), so that it becomes

$$x_{n+1} = mHH_1 \cdots H_n x_{-n}.$$

Also by setting $p = 2n + 1$ and $i = n + 1$ in (41), we get

$$x_{-n-1} = \frac{1}{m} H_{2n}H_{2n-1} \cdots H_n x_n.$$

The differential equations to which these are equivalent give values of

$\partial^{n+1}x/\partial v^{n+1}$ and $\partial^{n+1}x/\partial u^{n+1}$ in terms of the $2n+1$ or p quantities

$$\frac{\partial^n x}{\partial u^n}, \frac{\partial^n x}{\partial v^n}, \frac{\partial^{n-1}x}{\partial u^{n-1}}, \frac{\partial^{n-1}x}{\partial v^{n-1}}, \cdots, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, x.$$

All other derivatives of order $n+1$ may be obtained in terms of these same p quantities by differentiation of (1). Similarly, when the period is even, let $p = 2n$ and $i = n, n+1$, giving the two equations

$$x_n = mHH_1 \cdots H_{n-1}x_{-n},$$

$$x_{-n-1} = \frac{1}{m} H_{2n-1}H_{2n-2} \cdots H_{n-1}x_{n-1}.$$

By means of these equations and (1) all derivatives of the nth order but $\partial^n x/\partial u^n$, and all derivatives of higher orders may be expressed in terms of the $2n$ or p quantities

$$\frac{\partial^n x}{\partial u^n}, \frac{\partial^{n-1}x}{\partial u^{n-1}}, \frac{\partial^{n-1}x}{\partial v^{n-1}}, \cdots, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, x.$$

In either case we have a completely integrable system of equations which possesses but p independent solutions. From this result follows the theorem stated by Tzitzeica:

A sequence of Laplace of period p can exist in space of no higher order than $p-1$.

In particular we note:

The only nets of period three are planar nets.

It will be observed that the conditions (34) and (36) do not involve the constant m. Neither is it involved in the above discussion of the complete integrability of equations (1) and (37). Again, the equations themselves show us that m is a significant constant, that is, one which cannot be reduced to unity by any change of parameter. Then the solutions of the system, which are the coördinates x of our fundamental net N, may be written $x^i\ (u, v; m)$, $i = 1, 2, \cdots, p$.

If we replace m by another constant, m', so that we have the equation $x_p' = m'x'$, instead of (37), this equation forms with (1) another completely integrable system with p independent solutions, which we may call $x'\ (u, v; m')$. A similar set may be obtained for every value of the constant. We may state this result as follows:

If an equation of Laplace be the point equation of a net whose Laplace sequence is periodic of period p, it is the point equation of an infinity of nets having the same property.

3. **Levy sequences. The first Levy sequence.** In equation (3) of the introduction, functions ξ and η are defined as the coördinates of the Levy

transforms of N by means of a solution θ of the point equation (1). For the study of these transforms in connection with the Laplace sequence, it is advantageous to denote the coördinates by $x_{0,1}$ and $x_{-1,0}$, defined by

$$x_{0,1} = \frac{1}{\theta}\begin{vmatrix} \theta & x \\ \theta_1 & x_1 \end{vmatrix}, \qquad x_{-1,0} = \frac{1}{\theta_{-1}}\begin{vmatrix} \theta_{-1} & x_{-1} \\ \theta & x \end{vmatrix},$$

as they indicate by their form that the points given by any parameter values lie on the line joining the points of N and its Laplace transforms with the same parameters. The functions θ_1 and θ_{-1}, defined by

$$(44) \qquad \theta_1 = \frac{\partial\theta}{\partial v} - \frac{\partial \log a}{\partial v}\theta; \qquad \theta_{-1} = \frac{\partial\theta}{\partial u} - \frac{\partial \log b}{\partial u}\theta,$$

are called the first and minus first Laplace transforms of θ and are solutions of the point equation of N_1 and N_{-1} respectively. As we have the relations

$$\theta x_{0,1} = -\frac{\partial\theta}{\partial v}\xi, \qquad \theta_{-1}x_{-1,0} = \frac{\partial\theta}{\partial u}\eta,$$

the functions $x_{0,1}$ and $x_{-1,0}$ differ from ξ and η only by factors of proportionality, and consequently are coördinates of the Levy transforms.

We shall accordingly denote the Levy transforms by $N_{0,1}$ and $N_{-1,0}$; their point equations are denoted by

$$(45) \qquad \left[x_{0,1};\ \frac{a\theta_1}{\theta},\ b,\ \frac{b}{\theta}\right],$$

and

$$\left[x_{-1,0};\ \frac{a\theta}{\theta_{-1}},\ b,\ \frac{a}{\theta_{-1}}\right].$$

Let the net $N_{r,r+1}$ be defined by its coördinates $x_{r,r+1}$ namely,

$$(46) \qquad x_{r,r+1} = \frac{1}{\theta_r}\begin{vmatrix} \theta_r & x_r \\ \theta_{r+1} & x_{r+1} \end{vmatrix},$$

where r is any positive or negative integer or zero, and where θ_i is formed from θ by (21) and (24) as x_i is formed from x. The order of the subscripts in $N_{r,r+1}$ indicates that the points of these nets are to be considered as situated on the tangents to the curves u = constant of the net N_r; that is, on the line between any net N_r and its positive Laplace transform, N_{r+1}. The application of (23) and (26) to nets of this type leads to the theorem:

Any Laplace transform N_r of a net N has the nets $N_{r-1,r}$ and $N_{r,r+1}$ as its Levy transforms by means of θ_r, the rth Laplace transform of a solution θ of the point equation of N; or, $N_{r,r+1}$ is a Levy transform of N_r by means of θ_r, and of N_{r+1} by means of θ_{r+1}.

Let us express the coördinates of the first Laplace transform of $X_{-1,0}$, following (2) and simplify them. We find

$$\frac{\partial x_{-1,0}}{\partial v} - \frac{\partial}{\partial v}\log\frac{a\theta}{\theta_{-1}}\, x_{-1,0} = x_{0,1},$$

that is, *the Levy transforms of a net by means of the same solution of its point equation are Laplace transforms of one another.*

From the last two theorems $X_{r-1,r}$ and $X_{r,r+1}$ are Laplace transforms of one another for every value of r. Then $X_{r,r+1}$, $(r = \cdots, -2, -1, 0, 1, 2, \cdots)$, is a sequence of Laplace; it will be called the first Levy sequence. In the expressions for the point equations and the formulas for the partial derivatives of the coördinates of the nets of this sequence, it is necessary to distinguish between positive and negative subscripts. If r and s be positive integers, we have

$$\left[x_{r,r+1};\ \frac{aHH_1\cdots H_{r-1}\theta_{r+1}}{\theta_r},\ b,\ \frac{b^{r+1}}{a^rH^{r-1}H_1{}^{r-2}\cdots H_{r-2}\theta_r}\right];$$

$$\left[x_{-s-1,-s};\ \frac{a\theta_{-s}}{\theta_{-s-1}},\ bKK_{-1}\cdots K_{-s+1},\ \frac{a^{s+1}}{b^sK^{s-1}K_{-1}{}^{s-2}\cdots K_{-s+2}\theta_{-s-1}}\right];$$

$$\frac{\partial x_{r,r+1}}{\partial u} = H_{r-1,r}x_{r-1,r} + \frac{\partial \log b}{\partial u}x_{r,r+1},$$

$$\frac{\partial x_{r,r+1}}{\partial v} = \frac{\partial \log a_{r,r+1}}{\partial v}x_{r,r+1} + x_{r+1,r+2};$$

$$\frac{\partial x_{-s-1,-s}}{\partial u} = \frac{K_{-s-1,-s}}{K_{-s}}x_{-s-2,-s-1} + \frac{\partial \log b_{-s-1,-s}}{\partial u}x_{-s-1,-s},$$

$$\frac{\partial x_{-s-1,-s}}{\partial v} = \frac{\partial \log a_{-s-1,-s}}{\partial v}x_{-s-1,-s} + K_{-s+1}x_{-s,-s+1}.$$

4. **The second Levy sequence. Levy sequences of higher orders.** The first Levy sequence is built up from the fundamental sequence of Laplace by the use of a solution θ of (1) and its Laplace transforms. On this Levy sequence which is itself a sequence of Laplace, we may build a second Levy sequence and so on indefinitely.

Let $\theta_{0,1}$ be a solution of equation (45). The Levy transforms of $X_{0,1}$ by means of this solution are the nets $X_{0,2}$ and $X_{-1,1}$, whose coördinates are defined in accordance with (46) as follows,

$$x_{0,2} = \frac{1}{\theta_{0,1}}\begin{vmatrix}\theta_{0,1} & x_{0,1}\\ \theta_{1,2} & x_{1,2}\end{vmatrix}, \qquad x_{-1,1} = \frac{1}{\theta_{-1,0}}\begin{vmatrix}\theta_{-1,0} & x_{-1,0}\\ \theta_{0,1} & x_{0,1}\end{vmatrix},$$

where $\theta_{1,2}$ is the first Laplace transform of $\theta_{0,1}$, and $\theta_{-1,0}$ is its minus

first transform divided by $H_{-1,0}$, a quantity which occurs similarly in $x_{-1,0}$. The point equations of $N_{0,2}$ and $N_{-1,1}$ are denoted by

$$(47)\qquad \left[x_{0,2};\ \frac{a\theta_1\theta_{1,2}}{\theta\theta_{0,1}},\ b,\ \frac{b^2}{\theta\theta_{0,1}}\right], \qquad \left[x_{-1,1};\ \frac{a\theta\theta_{0,1}}{\theta_{-1}\theta_{-1,0}},\ b,\ \frac{ab}{\theta_{-1}\theta_{-1,0}}\right].$$

The same pair of theorems which established the first Levy sequence and the fact that it is a sequence of Laplace are valid here. We denote by $N_{r,r+2}$ and $N_{-s-2,-s}$ the general nets of the second Levy sequence and give their coördinates, namely

$$x_{r,r+2} = \frac{1}{\theta_{r,r+1}}\begin{vmatrix} \theta_{r,r+1} & x_{r,r+1} \\ \theta_{r+1,r+2} & x_{r+1,r+2} \end{vmatrix},$$

$$x_{-s-2,-s} = \frac{1}{\theta_{-s-2,-s-1}}\begin{vmatrix} \theta_{-s-2,-s-1} & x_{-s-2,-s-1} \\ \theta_{-s-1,-s} & x_{-s-1,-s} \end{vmatrix}.$$

Using the second Levy sequence and a solution of (47), a third Levy sequence may be formed. We shall not give the details of this sequence but pass at once to the kth or general sequence. Here the net corresponding to $N_{0,1}$ and $N_{0,2}$ is $N_{0,k}$. Its coördinates and point equation and the accompanying differentiation formulas can be written down by analogy with the corresponding expressions for $N_{0,1}$ and $N_{0,2}$, and their accuracy established by induction. Similar methods may be applied in the study of the other nets of the general sequence. The coördinates of the general net $N_{r,r+k}$ are defined by

$$x_{r,r+k} = \frac{1}{\theta_{r,r+k-1}}\begin{vmatrix} \theta_{r,r+k-1} & x_{r,r+k-1} \\ \theta_{r+1,r+k} & x_{r+1,r+k} \end{vmatrix},$$

where r is any positive or negative integer or zero, and k any positive integer.

In forming the second Levy sequence, we made use of a solution $\theta_{0,1}$ of equation (45); we now investigate the nature of this function. Suppose θ' to be a solution of (1) such that there is no linear relation connecting θ, θ' and the coördinates x. If

$$(52)\qquad \theta_{0,1} = \frac{1}{\theta}\begin{vmatrix} \theta & \theta' \\ \theta_1 & \theta_1' \end{vmatrix},$$

then $\theta_{0,1}$ is a solution of (45). It will now be proved that, conversely, to a solution $\theta_{0,1}$ of (45), not linearly dependent on the coördinates $x_{0,1}$, there corresponds a solution θ' of (1) linearly independent of the x's and of θ. Consider the net $N_{0,1}$ as the projection in $(n-1)$ space of a net $\bar{N}_{0,1}$ in n-space whose coördinates are $x_{0,1}^{(1)}$, $x_{0,1}^{(2)}$, $\cdots$, $x_{0,1}^{(n)}$, $\theta_{0,1}$. Then the congruence G, composed of the lines joining corresponding points

of N and N_1, is the projection of a congruence $\bar{G}$ in n-space conjugate to the net $\bar{N}_{0,1}$. One of the focal nets of this congruence, say $\bar{N}$, projects into the net N. Now the solutions $x^{(i)}$ of (1) are coördinates both of N and of $\bar{N}$ and, with θ, play the same rôle in both spaces in forming the coördinates $x_{0,1}{}^{(i)}$ of $N_{0,1}$ and $\bar{N}_{0,1}$. But in order to form the last coördinates of $\bar{N}_{0,1}$, namely $\theta_{0,1}$, there must be an $(n+1)$st coördinate of $\bar{N}$, a solution of (1) which may be called θ'.

Again the third Levy sequence depends on a solution $\theta_{0,2}$ of (47) for its formation. The argument of the last paragraph then demands as a necessary and sufficient condition for the existence of this solution a second solution $\theta'_{0,1}$ of (45) not linearly dependent on those already obtained. In the same manner, $\theta'_{0,1}$ calls for a third solution, say θ'', of (1) not linearly dependent on the solutions already used, such that

$$\theta'_{0,1} = \frac{1}{\theta}\begin{vmatrix} \theta & \theta'' \\ \theta_1 & \theta_1'' \end{vmatrix}.$$

The final effect of this argument is to base the kth or general sequence on k solutions, $\theta, \theta', \cdots, \theta^{(k-1)}$ of (1) such that there is no linear relation between them and the x's.

For further developments, we must prove, as a lemma, a property of determinants. Consider the general determinant of the nth order

$$D = |a_{lm}|, \qquad l, m = 1, 2, \cdots n.$$

Subtract from each element of the ith row the product of the corresponding element of the $(i-1)$st row by $a_{i,1}/a_{i-1,1}$, $(i = n, n-1, \cdots, 2,)$ and develop the result by the elements of the first column. We have

$$D = a_{1,1}\begin{vmatrix} \frac{1}{a_{1,1}}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} & \frac{1}{a_{1,1}}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} & \cdots & \frac{1}{a_{1,1}}\begin{vmatrix} a_{1,1} & a_{1,n} \\ a_{2,1} & a_{2,n} \end{vmatrix} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a_{n-1,1}}\begin{vmatrix} a_{n-1,1} & a_{n-1,2} \\ a_{n,1} & a_{n,2} \end{vmatrix} & & \cdots & \frac{1}{a_{n-1,1}}\begin{vmatrix} a_{n-1,1} & a_{n-1,n} \\ a_{n,1} & a_{n,n} \end{vmatrix} \end{vmatrix} = a_{1,1}\Delta$$

where Δ is of order $n-1$, so that

$$\Delta = \frac{1}{a_{11}} D. \tag{53}$$

The coördinates of $N_{0,2}$ are

$$x_{0,2} = \frac{1}{\theta_{0,1}}\begin{vmatrix} \theta_{0,1} & x_{0,1} \\ \theta_{1,2} & x_{1,2} \end{vmatrix}.$$

Using $r = 0, 1$ in (46) and the analogous expressions for $\theta_{0,1}$ and $\theta_{1,2}$ the coördinates $x_{0,2}$ become determinants of the form of Δ. On applying the

property expressed in (53) to them, we find

$$x_{0,\,2} = \frac{1}{\theta\theta_{0,\,1}}\begin{vmatrix} \theta & \theta' & x \\ \theta_1 & \theta_1' & x_1 \\ \theta_2 & \theta_2' & x_2 \end{vmatrix} \equiv \frac{1}{\theta\theta_{0,\,1}}\,|\,\theta\theta_1'x_2\,|,$$

the latter expression being an abbreviated form in which only the elements of the main diagonal are shown.

Consider $x_{0,\,k}$, the coördinates of the net $N_{0,\,k}$. By definition

$$x_{0,\,k} = \frac{1}{\theta_{0,\,k-1}}\begin{vmatrix} \theta_{0,\,k-1} & x_{0,\,k-1} \\ \theta_{1,\,k} & x_{1,\,k} \end{vmatrix}.$$

Then by (53)

$$x_{0,\,k} = \frac{1}{\theta_{0,\,k-2}\theta_{0,\,k-1}}\,|\,\theta_{0,\,k-2}\,\theta'_{1,\,k-1}\,x_{2,\,k}\,|,$$

and by its repeated use

$$x_{0,\,k} = \frac{1}{\theta\theta_{0,\,1}\theta_{0,\,2}\cdots\theta_{0,\,k-1}}\,|\,\theta\theta_1'\theta_2''\cdots\theta_{k-1}^{(k-1)}x_k\,|.$$

As this method of exhibiting the coördinates of the nets of the Levy sequences is a purely algebraic matter, we have at once,

$$(54)\qquad x_{r,\,r+k} = \frac{1}{\theta_r\theta_{r,\,r+1}\cdots\theta_{r,\,r+k-1}}\,|\,\theta_r\theta_{r+1}'\cdots\theta_{r+k-1}^{(k-1)}x_{r+k}\,|,$$

where r may be any positive or negative integer, or zero. We shall call the determinant in the above equation $X_{r,\,r+k}$; a determinant like it but for the last column, in which the Laplace transforms of x are replaced by those of $\theta^{(k)}$, a $(k+1)$st solution of (1), we shall call $\Theta_{r,\,r+k}$. Then

$$(55)\qquad \theta_{r,\,r+k} = \frac{1}{\theta_r\theta_{r,\,r+1}\cdots\theta_{r,\,r+k-1}}\Theta_{r,\,r+k}.$$

From (54) and (55), we have

$$(56)\qquad \theta_r\theta_{r,\,r+1}\cdots\theta_{r,\,r+k-1}x_{r,\,r+k} = X_{r,\,r+k},$$

and

$$(57)\qquad \theta_r\theta_{r,\,r+1}\cdots\theta_{r,\,r+k} = \Theta_{r,\,r+k}.$$

These equations are valid for any integral value of r, and for any positive integral value of k. If we replace k in (57) by $k-1$ and use the result in (56) and (57), we get

$$(58)\qquad \Theta_{r,\,r+k-1}x_{r,\,r+k} = X_{r,\,r+k}$$

and

$$\Theta_{r,\,r+k-1}\theta_{r,\,r+k} = \Theta_{r,\,r+k}.$$

Since the $X_{r,\,r+k}$ are proportional to the $x_{r,\,r+k}$, the former may serve equally well as homogeneous coördinates of the nets $N_{r,\,r+k}$.

In the preceding paragraph we have used solutions θ linearly independent of the coördinates x. The following theorem states the situation under the opposite condition.

If a solution θ of the equation (1) *used in the formation of any Levy sequence be linearly dependent on the coördinates x, all the nets of this Levy sequence lie in* ($\cancel{k}$ n − 2) *space; if i such solutions be used, in* ($\cancel{k}$ n − i − 1) *space.*

For, suppose $\theta = \sum_{i=1}^{i=n} g^{(i)} x^{(i)}$ where the $g^{(i)}$ are constants not all zero; then $\theta_k = \sum_{i=1}^{i=n} g^{(i)} x_k^{(i)}$ for every k. Now using these values of the Laplace transforms of θ, we have

$$\sum_{i=1}^{n} g^{(i)} X^{(i)}_{r,\, r+k} = \left| \theta_r \theta'_{r+1} \cdots \theta^{(k-1)}_{r+k-1} \Sigma g^{(i)} x_{r+k} \right| = 0,$$

since the first and last columns are identical, that is, the coördinates of all nets of the sequence $X_{r,\, r+k}$ satisfy the equation of the hyperplane $\sum_{i=1}^{i=n} g^{(i)} z^{(i)} = 0$, where the $z^{(i)}$ are current coördinates. We observe that if all the $g^{(i)}$ but one, say $g^{(j)}$, are zero, the nets lie in the coördinate hyperplane $x^{(j)} = 0$. If $\theta' = \sum_{i=1}^{i=n} h^{(i)} x^{(i)}$, then by the above argument, the nets of the sequence lie also in the hyperplane $\sum_{i=1}^{i=n} h^{(i)} z^{(i)} = 0$. Thus they lie in the intersection of two hyperplanes, or in space of order $\cancel{k}$ n − 3. This proof may be extended to the case stated in the theorem.

Consider the coördinates of the net $X_{-s,\, r}$; $r, s \geq 0$, in the form

$$X_{-s,\, r} = \left| \theta_{-s} \theta'_{-s+1} \cdots \theta^{(k-1)}_{r-1} x_r \right|,$$

where we have written only the elements of the main diagonal. For each of the Laplace transforms occurring in these determinants substitute their values as linear functions of the derivatives of the θ's and the x's from (21) and (24). By suitable operations on the rows, the determinants may then be reduced to the form

$$X_{-s,\, r} = \left| \frac{\partial^s \theta}{\partial u^s} \frac{\partial^{s-1} \theta'}{\partial u^{s-1}} \cdots \theta^{(s)} \cdots \frac{\partial^{r-1} \theta^{(k-1)}}{\partial v^{r-1}} \frac{\partial^r x}{\partial v^r} \right| .$$

In this form the identity of the $X_{-s,\, r}$ with the coördinates of the derived nets of higher order k as defined by Tzitzeica is obvious. There are $k + 1$ derived nets of order k; for example, the Levy transforms $X_{-1,\, 0}$ and $X_{0,\, 1}$ are the derived nets of the first order; the nets $X_{-2,\, 0}$, $X_{-1,\, 1}$, and $X_{0,\, 2}$ are the derived nets of order two, and so for higher orders. We note especially that $X_{-1,\, 1}$ is *the derived net* of X depending on θ and θ' in the restricted use of that term; X, in turn, is *the derivant net* of X. Extending this term, we say that X is a derivant net of all nets $X_{-s,\, r}$; $r, s \geq 0$.

5. **Periodic Levy sequences.** If a sequence of Laplace is of period p, and in $(p - 1)$ space, we shall now develop certain conditions under which its Levy sequences have this same period. If the first Levy sequence is

to be periodic, we must have

$$x_{p,\,p+1} = \lambda x_{0,\,1},$$

where λ is an undetermined factor of proportionality. By the use of (37) and the value of x_{p+1} found by differentiating (37) with respect to v, we obtain

$$\frac{m}{\theta_p}\begin{vmatrix} \theta_p & x \\ \theta_{p+1} & x_1 \end{vmatrix} = \frac{\lambda}{\theta}\begin{vmatrix} \theta & x \\ \theta_1 & x_1 \end{vmatrix},$$

that is,

$$x_1(m - \lambda) - x\left(\frac{m\theta_{p+1}}{\theta_p} - \frac{\lambda\theta_1}{\theta}\right) = 0.$$

Now there are at least three coördinates x, and accordingly the coefficients of x and x_1 must be zero. We have $m = \lambda$, and $\theta_{p+1}/\theta_p = \theta_1/\theta$, which, because of the definition of θ_{p+1} and θ_1, and equation (36), becomes

$$\frac{\partial}{\partial v}\log\frac{\theta_p}{\theta} = 0. \tag{59}$$

Let us now differentiate $x_{p,\,p+1} = mx_{0,\,1}$ with respect to u. By applying formulas already derived for such derivatives, we get

$$x_{p-1,\,p} = mx_{-1,\,0}.$$

In this case, using (37) and (38) and expanding, we have $H_{p-1}\theta_{p-1}/\theta_p = \theta_{-1}/\theta$. Then (2) and (23) give

$$\frac{\partial}{\partial u}\log\frac{\theta_p}{\theta} = 0. \tag{60}$$

In consequence of (59) and (60), we see that

$$\theta_p = C_1\theta,$$

where C_1 is a constant whose value is to be studied further. To show that under this condition the nets $N_{p-i,\,p-i+1}$ and $N_{-i,\,-i+1}$ are identical, consider the coördinates

$$x_{p-1,\,p-i+1} = \frac{1}{\theta_{p-i}}\begin{vmatrix} \theta_{p-i} & x_{p-i} \\ \theta_{p-i+1} & x_{p-i+1} \end{vmatrix}.$$

Since (41) and (42) are true for θ if m be replaced by C_1, we have

$$x_{p-i,\,p-i+1} = mHH_1 \cdots H_{p-i}x_{-i,\,-i+1}.$$

For the second Levy sequence also to be periodic, it is necessary and sufficient that $\theta_{0,\,1}$ the solution of (45) by which the second Levy sequence is formed from the first, shall be such that

$$\theta_{p,\,p+1} = C_2\theta_{0,\,1}.$$

By referring to equation (52), we see that this will be the case if $\theta_p' = C_1'\theta'$. In general, we conclude that if the $(k-1)$st sequence is periodic, the necessary and sufficient condition for the kth sequence to be periodic, is that

$$\theta_{p,\ p+k-1} = C_k\theta_{0,\ k-1}$$

and that this condition will be fulfilled if $\theta, \theta', \cdots \theta^{(k-1)}$ are such that their pth Laplace transforms are constant multiples of them. Evidently under these last conditions not only is the kth Levy sequence periodic, but also all the sequences of order less than k.

The disposition of the constant multipliers in various ways leads to some interesting results. By the last theorem of section 2, there are p solutions θ for every value of the constant occurring in (37). First, let us suppose that C_1 is equal to m. Then θ must be a linear combination of the x's and therefore the theorem of section 4 applies and the nets of this periodic Levy sequence lie space of order $p-2$. This result was noted by Tzitzeica. For sequences of higher orders, it may be generalized into the following theorem:

If a sequence of Laplace of period p lie in $(p-1)$ space, and has coördinates such that $x_p = mx$, and if a Levy sequence of order k, periodic or not, based on this sequence of Laplace, be formed by the use of k solutions θ of the original point equation, of which one is such that $\theta_p = m\theta$, the nets of this Levy sequence lie in space of order $p-2$; if i such solutions be used, the Levy sequence is in space of order $p-i-1$.

To prove this theorem, we need first to recall that there are but p solutions of the system of partial differential equations satisfied by the coördinates of a periodic sequence of Laplace; therefore, if $\theta_p = m\theta$, θ is a linear function of the coördinates x. The proof is then completed by the application of the last theorem of section 4.

Consider now the Levy sequences which can be formed on a periodic sequence of Laplace by the use of the set of p solutions $\theta, \theta', \cdots, \theta^{(p-1)}$ such that $\theta_p^{(i)} = m'\theta^{(i)}$, $m' \neq m$. There will be p periodic first Levy sequences, $p(p-1)/2$ periodic second Levy sequences, in general, as many of the kth order as the number of combinations of p things taken k at a time and finally, one periodic sequence of the pth order. It will now be shown that this pth sequence coincides with the original sequence of Laplace. For, consider the coördinates of the net $X_{0,\ p}$, namely, $X_{0,\ p}$. We have

$$X_{0,\ p} = \begin{vmatrix} \theta & \theta' & \cdots & \theta^{(p-1)} & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta_p & \theta_p' & \cdots & \theta_p^{(p-1)} & x_p \end{vmatrix} = \begin{vmatrix} \theta & \theta' & \cdots & \theta^{(p-1)} & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m'\theta & m'\theta' & \cdots & m'\theta^{(p-1)} & mx \end{vmatrix}.$$

Subtracting m' times the first row from the last, then

$$X_{0,\,p} = (m - m')\Theta_{0,\,p-1}x,$$

so that the coördinates $X_{0,\,p}$ are proportional to the x's. In general the coördinates $X_{r,\,r+p}$ of the net $N_{r,\,r+p}$ are determinants such that in each the elements of its last row may be made all zero but the last, which will be a constant multiple of x_r. Therefore $X_{r,\,r+p}$ is proportional to x_r, and the nets $N_{r,\,r+p}$ coincide with the original Laplace sequence.

6. **Nets in relation T and their Laplace transforms.** In the introduction a geometric definition of the relation T was given; Eisenhart has shown that, if $\bar{N}$ be a net in relation T with N, their analytic relation is expressed in the statement that the homogeneous coördinates $\bar{x}$ of $\bar{N}$ may be obtained from quadratures of the form

$$(61) \qquad \frac{\partial \bar{x}}{\partial u} = \tau \frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial \bar{x}}{\partial v} = \sigma \frac{\partial}{\partial v}\left(\frac{x}{\theta}\right),$$

where θ is a solution of (1) different from any x. The factors τ and σ are not entirely arbitrary for the conditions of integrability of (61) show that they must be solutions of the equations

$$(62) \qquad \frac{\partial \tau}{\partial v} = (\sigma - \tau)\frac{\partial}{\partial v}\log\frac{a}{\theta}, \qquad \frac{\partial \sigma}{\partial u} = (\tau - \sigma)\frac{\partial}{\partial u}\log\frac{b}{\theta},$$

or their equivalents,

$$\frac{\partial}{\partial v}\log\frac{a\tau}{\theta} = \frac{\sigma}{\tau}\frac{\partial}{\partial v}\log\frac{a}{\theta}, \qquad \frac{\partial}{\partial u}\log\frac{b\sigma}{\theta} = \frac{\tau}{\sigma}\frac{\partial}{\partial u}\log\frac{b}{\theta}.$$

In connection with the derivation of the integrability conditions of (61) it is readily shown that the net $\bar{N}$ has the point equation

$$\frac{\partial^2 \bar{x}}{\partial u \partial v} = \frac{\partial}{\partial v}\log\frac{a\tau}{\theta}\frac{\partial \bar{x}}{\partial u} + \frac{\partial}{\partial u}\log\frac{b\sigma}{\theta}\frac{\partial \bar{x}}{\partial v}.$$

For an equation of this special type in which the term involving $\bar{x}$ is missing we shall use a symbol similar to (13) except that the last of the quantities within the brackets is omitted to indicate that the term in $\bar{x}$ is lacking. Thus, the point equation of $\bar{N}$ will be denoted by

$$(63) \qquad \left[\bar{x};\ \frac{a\tau}{\theta},\ \frac{b\sigma}{\theta}\right].$$

Using the fact that θ is a solution of (1), the invariants of $\bar{N}$ have the values

$$(64) \qquad \bar{H} = H - \frac{\partial^2 \log \tau}{\partial u \partial v}, \qquad \bar{K} = K - \frac{\partial^2 \log \sigma}{\partial u \partial v}.$$

From these developments, it appears that the determination of a net in relation T with N depends on a solution of (1), a pair of functions τ and σ which satisfy (62), and the quadratures (61). Eisenhart has shown that the problem may be given another aspect by the introduction of a function ϕ, defined by the equation

$$\tau - \sigma = \phi\theta. \tag{65}$$

By differentiation and the use of (62) and (1) it may be shown that ϕ is a solution of the equation denoted by

$$\left[\phi;\ \frac{1}{a},\ \frac{1}{b},\ \frac{1}{ab}\right]. \tag{66}$$

But this is the adjoint of (1). Accordingly, the problem is reduced to the finding of a solution of (1) and a solution of its adjoint equation, and two sets of quadratures, namely

$$\begin{aligned} \frac{\partial\tau}{\partial u} &= \phi\theta\,\frac{\partial \log b\phi}{\partial u}, & \frac{\partial\tau}{\partial v} &= -\,\phi\theta\,\frac{\partial}{\partial v}\log\frac{a}{\theta}, \\ \frac{\partial\sigma}{\partial u} &= \phi\theta\,\frac{\partial}{\partial u}\log\frac{b}{\theta}, & \frac{\partial\sigma}{\partial v} &= -\,\phi\theta\,\frac{\partial \log a\phi}{\partial v}, \end{aligned} \tag{67}$$

which follow from (65) and (62), and (61).

A discussion of the effect on the net $\bar{N}$ of the arbitrary constants arising from these quadratures is in order at this time. If $\bar{x}$ be the coördinates of the net $\bar{N}$ when the additive constant c to τ and σ is set equal to zero, then for any other value of c, the coördinates of the T transform become $\bar{x} + cx/\theta$. This point is on the line joining corresponding points of N and $\bar{N}$. Consequently, we may say that the variation of this constant leaves the conjugate congruence of the transformation unchanged but moves the points of the net along the lines of this congruence.

Again if $\bar{x}^{(i)}$ and $\bar{x}^{(i)} + c_i$ are the coördinates of nets obtained by different values of the constant of integration in (61), the line of intersection of the tangent planes to the nets is the same for all values of c_i. This is a result of equation (61) since the coördinates of the Levy transforms of N by means of θ may be taken as $\partial(x/\theta)/\partial u$ and $\partial(x/\theta)/\partial v$. The totality of such lines of intersection, or the joins of corresponding points of the Levy transforms form a congruence which has been termed by Guichard* the harmonic congruence of the transformation. We may say then, that the variation of the constant arising from (61) leaves the harmonic congruence of the transformation unchanged. Conversely, all nets harmonic to this congruence are so determined, since it has been

* Guichard, Annales de l'École Normale Sup., 3e Série, t. 14 (1897), p. 483.

shown by Eisenhart* that two nets harmonic to a congruence are in relation T.

Now if x_1 and θ_1 be the first Laplace transforms of x and θ, the coördinates of $\bar{N}_1$, a T transform of N_1 will be given by quadratures similar to (61), namely,

$$\frac{\partial \bar{x}_1}{\partial u} = \tau_1 \frac{\partial}{\partial u}\left(\frac{x_1}{\theta_1}\right), \qquad \frac{\partial \bar{x}_1}{\partial v} = \sigma_1 \frac{\partial}{\partial v}\left(\frac{x_1}{\theta_1}\right). \tag{68}$$

The integrability conditions of this quadrature give equations for τ_1 and σ_1 analogous to (62),

$$\frac{\partial \tau_1}{\partial v} = (\sigma_1 - \tau_1)\frac{\partial}{\partial v}\log\frac{aH}{\theta_1}, \qquad \frac{\partial \sigma_1}{\partial u} = (\tau_1 - \sigma_1)\frac{\partial}{\partial u}\log\frac{b}{\theta_1}$$

and we also find that the point equation of $\bar{N}_1$ is denoted by

$$\left[\bar{x}_1;\ \frac{aH\tau_1}{\theta_1},\ \frac{b\sigma_1}{\theta_1}\right]. \tag{69}$$

In order to determine the relation between τ_1, σ_1 and τ, σ, we proceed as follows. If $\bar{N}_1$ is the first Laplace transform of $\bar{N}$, the invariants $\bar{H}_1$ and $\bar{H}$ of these nets should be related as are the invariants H_1 and H in (12). Forming the corresponding relation, we have

$$\bar{H}_1 = -\frac{\partial^2}{\partial u \partial v}\log\frac{a\tau\bar{H}}{b\sigma} + \bar{H}.$$

On reducing this equation by the use of (69), (64), and (12), we find that

$$\frac{\partial^2}{\partial u \partial v}\log\tau_1 = \frac{\partial^2}{\partial u \partial v}\log\frac{\tau^2\bar{H}}{\sigma H}. \tag{70}$$

Similar reckoning performed with $\bar{K}_1$ and $\bar{K}$ shows that

$$\frac{\partial^2 \log\sigma_1}{\partial u \partial v} = \frac{\partial^2 \log\tau}{\partial u \partial v}. \tag{71}$$

Now the equations denoted by (63) and (69) are of the form which must be satisfied by the non-homogeneous coördinates of a net, and this suggests that the same relation may hold between the coördinates of $\bar{N}_1$ and $\bar{N}$ that holds in the non-homogeneous case. This is shown to be true, for if we substitute

$$\bar{x}_1 = \bar{x} - \frac{1}{\frac{\partial}{\partial v}\log\frac{a\tau}{\theta}}\frac{\partial \bar{x}}{\partial v},$$

* A result as yet unpublished.

and

$$\tau_1 = \frac{\tau^2 \bar{H}}{\sigma H}, \qquad \sigma_1 = \tau,$$

—particular solutions of (70) and (71)—and the values of x_1 and θ_1 given by (2) and (44) in equations (68), they are identically true. We have proved then that the T transforms of a net N and its first Laplace transform whose coördinates are obtained from the quadratures (61) and (68), where θ_1 is the first Laplace transform of θ, and where τ_1 and σ_1 have the above values, are Laplace transforms of one another.

We find that the difference $\tau_1 - \sigma_1$, when reduced by the use of (64) and (67), is equal to $-\phi_{-1}\,\theta_1/H$, where

$$\phi_{-1} = \frac{\partial \phi}{\partial u} + \frac{\partial \log b}{\partial u}\phi,$$

following (2) and (66). Now ϕ_{-1} satisfies the equation denoted by

$$(72) \qquad \left[\phi_{-1};\frac{1}{a},\frac{H}{b},\frac{1}{a^2}\right].$$

and ϕ_{-1}/H, because of (17), satisfies

$$\left[\frac{\phi_{-1}}{H};\frac{1}{aH},\frac{1}{b},\frac{1}{a^2H}\right].$$

But this equation is the adjoint of (16); so that we have the net $\bar{N}_1$ based on a solution of the adjoint of the point equation of N_1 which is proportional to the minus first Laplace transform of ϕ.

In general, we have that, if $\bar{N}_r$ is a T transform of N_r, the rth Laplace transform of N, whose coördinates are given by the quadratures

$$(73) \qquad \frac{\partial \bar{x}_r}{\partial u} = \tau_r \frac{\partial}{\partial u}\left(\frac{x_r}{\theta_r}\right), \qquad \frac{\partial \bar{x}_r}{\partial v} = \sigma_r \frac{\partial}{\partial v}\left(\frac{x_r}{\theta_r}\right),$$

then the nets $\bar{N}_r$ $(r = 0, 1, 2, \cdots)$, form a sequence of Laplace. In this general case, we find, as in the particular cases we have considered, that ϕ'_{-r}, defined by

$$\phi'_{-r} = \frac{\tau_r - \sigma_r}{\theta_r},$$

is a solution of the adjoint of the point equation of N_r, which is denoted by

$$(74) \qquad \left[\phi'_{-r};\frac{1}{aHH_1\cdots H_{r-1}},\frac{1}{b},\frac{b^{r-1}}{a^{r+1}H^rH_1^{r-1}\cdots H_{r-1}}\right].$$

But using equation (25) we may denote the equation of Laplace satisfied

by ϕ_{-r}, defined as

$$\phi_{-r} = \frac{\partial \phi_{-r+1}}{\partial u} + \frac{\partial}{\partial u} \log \frac{b}{HH_1 \cdots H_{r-1}} \phi_{-r+1}$$

by

$$\left[\phi_{-r};\ \frac{1}{a},\ \frac{HH_1 \cdots H_{r-1}}{b},\ \frac{b^{r-1}}{a^{r+1} H^{r-1} \cdots H_{r-2}} \right]. \tag{75}$$

But the quantity $HH_1 \cdots H_{r-1}\phi'_{-r}$, because of (74) and (17) also satisfies this equation, and therefore we have

$$\frac{\tau_r - \sigma_r}{\theta_r} = \phi'_{-r} = \frac{\phi_{-r}}{HH_1 \cdots H_{r-1}}.$$

Therefore the quadratures to determine τ_r and σ_r, corresponding to (67) are

$$\begin{aligned}
\frac{\partial \tau_r}{\partial u} &= \frac{\theta_r \phi_{-r}}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial u} \log \frac{b\phi_{-r}}{HH_1 \cdots H_{r-1}}, \\
\frac{\partial \tau_r}{\partial v} &= - \frac{\theta_r \phi_r}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial v} \log \frac{aH \cdots H_{r-1}}{\theta_r}, \\
\frac{\partial \sigma_r}{\partial u} &= \frac{\theta_r \phi_{-r}}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial u} \log \frac{b}{\theta_r}, \\
\frac{\partial \sigma_r}{\partial v} &= - \frac{\theta_r \phi_{-r}}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial v} \log a\phi_{-r}.
\end{aligned} \tag{76}$$

7. **Periodic sequences of T transforms.** As a preliminary step in the question of the periodicity of the T transforms, we investigate the adjoint equation of (1) when (34) and (36) are satisfied. These equations are necessary and sufficient conditions for a periodic sequence of Laplace whose coördinates satisfy (1) and (37). Now if the invariants of (66) are formed, it is found that they are the same as H and K but are interchanged. Similarly the invariants of (72) are those of (16), that is, H_1 and K_1, but interchanged; and in general, the invariants of (75) are H_r and K_r interchanged. We also notice that in working with (72), a and b are replaced by $1/a$ and $1/b$, respectively. Then the conditions on the coefficients of (66) which assure solutions such that

$$\phi = n\phi_{-p}$$

are equivalent to (34) and (36), and we may state the following theorem:

If an equation of Laplace has periodic solutions, so has its adjoint.

Suppose that the fundamental sequence is periodic of period p, and that $x_p = mx$, the conditions (34) and (36) being satisfied. Let ϕ, the solution of the adjoint equation of (1) which determines the quadratures

(67) and (61), be such that $\phi = n\phi_{-p}$; also let the solution θ of (1) involved in these quadratures be such that $\theta_p = m'\theta$. Then if we set $r = p$ the quadratures (73) and (76) which determine a pth Laplace transform of $\overline{N}$ become

$$(79)\qquad \begin{aligned} \frac{\partial \tau_p}{\partial u} &= \frac{m'}{n}\theta\phi\frac{\partial}{\partial u}\log b\phi, & \frac{\partial \tau_p}{\partial v} &= -\frac{m'}{n}\theta\phi\frac{\partial}{\partial v}\log\frac{a}{\theta}, \\ \frac{\partial \sigma_p}{\partial u} &= \frac{m'}{n}\theta\phi\frac{\partial}{\partial u}\log\frac{b}{\theta}, & \frac{\partial \sigma_p}{\partial v} &= -\frac{m'}{n}\theta\phi\frac{\partial}{\partial v}\log a\phi, \end{aligned}$$

and

$$(80)\qquad \frac{\partial \bar{x}_p}{\partial u} = \frac{m}{n}\tau\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial \bar{x}_p}{\partial v} = \frac{m}{n}\sigma\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right),$$

which differ from (67) and (61) only by constant factors. Then the coördinates $\bar{x}_p$ can differ from $\bar{x}$ only by a constant factor arising from the factors appearing in (79) and (80), or by an additive constant from the final quadratures. But these additive constants are entirely arbitrary, and since we are dealing with homogeneous coördinates the factor is immaterial.

We have, now, the following theorem: *Let* (1) *be the point equation of the fundamental net N of a sequence of Laplace of period p, whose coördinates x are such that $x_p = mx$; if θ be a solution of* (1) *such that $\theta_p = m'\theta$, and if ϕ be a solution of the adjoint equation of* (1) *such that $\phi = n\phi_{-p}$, then each T transform of N determined by quadratures from θ and ϕ is the fundamental net of a sequence of Laplace of period p; moreover, each of the nets of these sequences is a T transform of the corresponding net of the original sequence.*

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